

CURRENT VALUED MEASURES AND GEÖCZE AREA⁽¹⁾

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Abstract. If f is a continuous mapping of finite Geöcze area from a polyhedral region $X \subset R^k$ into R^n , $2 \leq k \leq n$, then, under suitable hypotheses, one can associate with f , by means of the Cesari-Weierstrass integral, a current valued measure T over the middle space of f . In particular, if either $k=2$ or the $k+1$ -dimensional Hausdorff measure of $f(X)$ is zero, then T is essentially the same as a current valued measure defined by H. Federer and hence serves to describe the tangential properties of f and the multiplicities with which f assumes its values. Further, the total variation of T is equal to the Geöcze area of f .

1. Introduction. Suppose f is a continuous mapping of finite Geöcze area, $V(f)$, from a polyhedral region $X \subset R^k$ into R^n , $2 \leq k \leq n$. If f belongs to the class $\mathcal{T}^*(k, n)$ defined by T. Nishiura [13], then (Theorem 1) we can associate with f , by means of the Cesari-Weierstrass integral, a current valued measure T over the middle space of f .

Suppose $\{f_i\}$ is a sequence of quasi-linear maps of X into R^n converging uniformly to f with bounded areas and let $f=l \circ m$ be the monotone-light factorization of f with middle space M . With each f_i we associate a current valued measure T_i over M defined by letting

$$T_i(g)(\varphi) = f_{i\#}[X \wedge (g \circ m)](\varphi) = \int_X (g \circ m)f_i^\#(\varphi)$$

whenever g is a continuous real valued function on M and φ is a differential k -form of class ∞ on R^n .

If $f \in \mathcal{T}^*(k, n)$ and there is a sequence $\{f_i\}$ of quasi-linear maps as above such that the sequence $\{T_i\}$ converges weakly to T , then we show in Theorem 3 that T is essentially the same as the current valued measure considered by H. Federer [7] and, in particular, shares its representation as the indefinite integral with respect to k -dimensional Hausdorff measure over M of a k -vector valued density v which describes the tangential properties of f and the multiplicity with which f assumes its values. Further, the total variation measure $\|T\|$ of T , taken with respect to mass, is equal to the Geöcze area measure μ induced on M by f .

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Let $\Lambda(k, n)$ denote the set of all k -tuples $\lambda = (\lambda_1, \dots, \lambda_k)$ of integers such that $1 \leq \lambda_1 < \lambda_2 < \dots < \lambda_k \leq n$. With each $\lambda \in \Lambda(k, n)$ associate the projection $p^\lambda: R^n \rightarrow R^k$ defined by

$$p^\lambda(y) = (y_{\lambda_1}, \dots, y_{\lambda_k}) \quad \text{for } y = (y_1, \dots, y_n) \in R^n.$$

In Theorem 4 we show that, if $f \in \mathcal{F}^*(k, n)$ and $\{f_i\}$ is any sequence of quasi-linear maps of X into R^n converging uniformly to f and such that

$$V(p^\lambda \circ f) = \lim_{i \rightarrow \infty} V(p^\lambda \circ f_i) \quad \text{for } \lambda \in \Lambda(k, n),$$

then the sequence $\{T_i\}$ of associated current valued measures converges weakly to T .

If $V(f) < \infty$ and either $k=2$ or $H_n^{k+1}(f(X))=0$, where H_n^m is m -dimensional Hausdorff measure in R^n , we note (Theorem 5) that the hypotheses of Theorem 3 are satisfied. In a subsequent paper we will show that, if either $k=2$ or $H_n^{k+1}(f(X))=0$, then the Lebesgue area and the Geöcze area of f coincide. From this one infers readily that T coincides with the current valued measure defined in [10] and that Theorem 4 represents a strengthening of the main result of [10].

In Theorem 2 we show that the current valued measure T associated with $f \in \mathcal{F}^*(k, n)$ possesses a representation as the indefinite integral with respect to Geöcze area measure over M of a k -vector valued function θ . In case either $k=2$ or $H_n^{k+1}(f(X))=0$, we show in Theorem 7 that $\theta(z) = v(z)/|v(z)|$ for μ almost every $z \in M$. Here $|v(z)|$ denotes the Euclidean norm of the k -vector $v(z)$.

2. The current valued measure T . Suppose k and n are integers, $2 \leq k \leq n$. Let $\Lambda(k, n)$ denote the set of all k -tuples $\lambda = (\lambda_1, \dots, \lambda_k)$ of integers such that $1 \leq \lambda_1 < \dots < \lambda_k \leq n$. Let $e_1, \dots, e_n$ be the usual basis in R^n . Then, denoting exterior multiplication by $\wedge$, the k -vectors $e_\lambda = e_{\lambda_1} \wedge \dots \wedge e_{\lambda_k}$, $\lambda \in \Lambda(k, n)$, form the usual basis for the space $\bigwedge_k(R^n)$ of k -vectors in R^n .

For each $\lambda \in \Lambda(k, n)$, let $p^\lambda: R^n \rightarrow R^k$ be defined by

$$p^\lambda(y) = (y_{\lambda_1}, \dots, y_{\lambda_k}) \quad \text{for } y = (y_1, \dots, y_n) \in R^n.$$

Consider each p^λ as projecting R^n onto the k -dimensional coordinate hyperspace of R^n determined by $e_{\lambda_1}, \dots, e_{\lambda_k}$.

Suppose f is a continuous mapping of finite Geöcze area $V(f)$ from a polyhedral region $X \subset R^k$ into R^n . With each simple polyhedral region $\pi \subset X$ we associate the k -vector

$$u(f, \pi) = \sum_{\lambda \in \Lambda(k, n)} u(f^\lambda, \pi) e_\lambda$$

where $u(f^\lambda, \pi) = \int_{R^k} O(f^\lambda, \pi, y) dy$ for $\lambda \in \Lambda(k, n)$. Here $f^\lambda = p^\lambda \circ f$ and $O(f^\lambda, \pi, y)$ denotes the topological index of $y \in R^k$ with respect to the mapping $f^\lambda|_\pi: \pi \rightarrow R^k$ if $y \in R^k - f^\lambda(\text{Bdry } \pi)$ and $O(f^\lambda, \lambda, y) = 0$ if $y \in f^\lambda(\text{Bdry } \pi)$.

Let $\mathcal{G}$ denote the set of all finite collections G of nonoverlapping simple polyhedral regions $\pi \subset X$. Define on $\mathcal{G}$ a real valued function δ by

$$\delta(G) = \max \{ \text{diam } f(\pi) : \pi \in G \} + \max \left\{ V(f^\wedge) - \sum_{\pi \in G} |u(f^\wedge, \pi)| : \lambda \in \Lambda(k, n) \right\}.$$

If $f \in \mathcal{F}^*(k, n)$, that is, if $\inf \{ \delta(G) : G \in \mathcal{G} \} = 0$, then, according to [13, §5.9], $u(f, \pi)$ is quasi-additive with respect to δ and mass $(\|\cdot\|)$ in $\bigwedge_k(R^n)$.

Let $f = l \circ m$ denote the monotone-light factorization of f with middle space M . Let d_f denote the usual metric induced on M by f and let $C(M)$ denote the space of continuous real valued functions on M .

THEOREM 1. *If $f \in \mathcal{F}^*(k, n)$, then, for each $g \in C(M)$ and continuous differential k -form φ in R^n ,*

$$(*) \quad T(g)(\varphi) = \lim_{\delta(G) \rightarrow 0} \sum_{\pi \in G} \frac{1}{|\pi|} \int_{\pi} g(m(x)) \varphi(f(x)) \cdot u(f, \pi) \, dx$$

exists. Here $|\pi|$ denotes k -dimensional Lebesgue measure of π and $\varphi(f(x)) \cdot u(f, \pi)$ denotes evaluation of the k -covector $\varphi(f(x))$ at $u(f, \pi)$.

Proof. The essential elements of a proof of the above statement can be found in [13] and [3]. We will give a complete proof here in our present notation.

Suppose $g \in C(M)$, φ is a continuous k -form on R^n , and $\varepsilon > 0$.

For any pair of simple polyhedral regions $\pi', \pi \subset X$ let $s(\pi', \pi) = 1$ if $\pi' \subset \pi$ and $s(\pi', \pi) = 0$ otherwise. According to [13, §5.9] there is a $\delta > 0$ such that, if $G \in \mathcal{G}$ with $\delta(G) < \delta$, then there is a $\gamma > 0$ such that

$$\sum_{\pi \in G} \left\| u(f, \pi) - \sum_{\pi_1 \in G_1} s(\pi_1, \pi) u(f, \pi_1) \right\| < \varepsilon$$

and

$$\sum_{\pi_1 \in G_1} \left[1 - \sum_{\pi \in G} s(\pi_1, \pi) \right] \|u(f, \pi_1)\| < \varepsilon$$

whenever $G_1 \in \mathcal{G}$ with $\delta(G_1) < \gamma$.

Suppose G and G_1 are as above. Since, for any simple polyhedral region $\pi \subset X$, the diameter of $m(\pi)$ relative to d_f does not exceed the diameter of $f(\pi)$, we can assume that $\delta(G)$ is so small that

$$\sup \{ \|g(m(x))\varphi(f(x)) - g(m(x'))\varphi(f(x'))\| : x, x' \in \pi \} < \varepsilon$$

for all $\pi \in G$. Here $\|\cdot\|$ denotes comass in the space $\bigwedge^k(R^n)$ of k -covectors in R^n .

Letting

$$M(g) = \sup \{ |g(z)| : z \in M \}$$

and

$$M(\varphi \circ f) = \sup \{ \|\varphi(f(x))\| : x \in X \},$$

we have

$$\begin{aligned}
 & \left| \sum_{\pi \in G} \frac{1}{|\pi|} \int_{\pi} g(m(x))\varphi(f(x)) \cdot u(f, \pi) \, dx - \sum_{\pi_1 \in G_1} \frac{1}{|\pi_1|} \int_{\pi_1} g(m(x))\varphi(f(x)) \cdot u(f, \pi_1) \, dx \right| \\
 & \leq \sum_{\pi \in G} \left| \frac{1}{|\pi|} \int_{\pi} g(m(x))\varphi(f(x)) \cdot u(f, \pi) \, dx \right. \\
 & \quad \left. - \sum_{\pi_1 \in G_1} \frac{s(\pi_1, \pi)}{|\pi_1|} \int_{\pi_1} g(m(x))\varphi(f(x)) \cdot u(f, \pi_1) \, dx \right| \\
 & \quad + \sum_{\pi_1 \in G_1} \left[1 - \sum_{\pi \in G} s(\pi_1, \pi) \right] \left| \frac{1}{|\pi_1|} \int_{\pi_1} g(m(x))\varphi(f(x)) \cdot u(f, \pi_1) \, dx \right| \\
 & \leq \sum_{\pi \in G} \left| \frac{1}{|\pi|} \int_{\pi} g(m(x))\varphi(f(x)) \cdot \left\{ u(f, \pi) - \sum_{\pi_1 \in G_1} s(\pi_1, \pi) u(f, \pi_1) \right\} \, dx \right| \\
 & \quad + \sum_{\pi \in G} \sum_{\pi_1 \in G_1} s(\pi_1, \pi) \left| \frac{1}{|\pi|} \int_{\pi} g(m(x))\varphi(f(x)) \cdot u(f, \pi_1) \, dx \right. \\
 & \quad \left. - \frac{1}{|\pi_1|} \int_{\pi_1} g(m(x))\varphi(f(x)) \cdot u(f, \pi_1) \, dx \right| \\
 & \quad + M(g)M(\varphi \circ f) \sum_{\pi_1 \in G_1} \left[1 - \sum_{\pi \in G} s(\pi_1, \pi) \right] \|u(f, \pi_1)\| \\
 & \leq M(g)M(\varphi \circ f) \sum_{\pi \in G} \left\| u(f, \pi) - \sum_{\pi_1 \in G_1} s(\pi_1, \pi) u(f, \pi_1) \right\| \\
 & \quad + \varepsilon \sum_{\pi_1 \in G_1} \|u(f, \pi_1)\| + \varepsilon M(g)M(\varphi \circ f) \\
 & \leq \left[2M(g)M(\varphi \circ f) + \binom{n}{k} V(f) \right] \varepsilon,
 \end{aligned}$$

because

$$\sum_{\pi_1 \in G_1} \|u(f, \pi_1)\| \leq \binom{n}{k} \sum_{\pi_1 \in G_1} |u(f, \pi_1)| \leq \binom{n}{k} V(f).$$

Since the above holds for any $G_1 \in \mathcal{G}$ with $\delta(G_1) < \gamma$, the theorem follows.

The formula (*) above defines a linear mapping of $C(M)$ into the space $E_k(R^n)$ of k -dimensional currents in R^n . (The notation concerning forms and currents is that of [9].)

Since, for $g \in C(M)$ and $\varphi \in E^k(R^n)$,

$$|T(g)(\varphi)| \leq \binom{n}{k} M(g)M(\varphi \circ f)V(f),$$

this mapping possesses a unique extension, also denoted by T , to the class of all bounded Borel measurable real valued functions on M such that Lebesgue's bounded convergence theorem holds. In particular, one obtains a countably additive current valued function T on the class of all Borel subsets of M which will be referred to as the current valued measure associated with f .

Assume throughout the remainder of this section that $f \in \mathcal{T}^*(k, n)$ so that the current valued measure T is defined.

For each simple polyhedral region $\pi \subset X$ and $\lambda \in \Lambda(k, n)$ let

$$u^+(f^\lambda, \pi) = \frac{1}{2}[|u(f^\lambda, \pi)| + u(f^\lambda, \pi)]$$

and

$$u^-(f^\lambda, \pi) = \frac{1}{2}[|u(f^\lambda, \pi)| - u(f^\lambda, \pi)].$$

For each set U open in X let

$$V^\pm(f^\lambda|U) = \sup \sum_{\pi \in \mathcal{S}} u^\pm(f^\lambda, \pi)$$

where the supremum is taken over all finite collections, $\mathcal{S}$, of nonoverlapping simple polyhedral regions $\pi \subset U$.

According to [13, §6] the function defined for all Borel sets $B \subset M$ by

$$\mu(B) = \inf \{V(f|m^{-1}(A)) : A \text{ open in } M \text{ and } A \supset B\}$$

is a finite Borel measure over M , and finite Borel measures $\mu^\lambda, \mu_\pm^\lambda$ over M can be defined analogously using the functions $V(f^\lambda|\cdot)$ and $V^\pm(f^\lambda|\cdot)$.

We note from [13, §6] that

$$\begin{aligned} \mu(M) &= V(f) = \lim_{\delta(G) \rightarrow 0} \sum_{\pi \in G} |u(f, \pi)|, \\ \mu^\lambda &= \mu_+^\lambda + \mu_-^\lambda \quad \text{for } \lambda \in \Lambda(k, n), \quad \mu(m(\text{Bdry } X)) = 0 \end{aligned}$$

and that

$$\mu(m(\pi)) = \mu([m(\pi)]^\circ) = V(f|\pi)$$

for each simple polyhedral region $\pi \subset X$. Here $[m(\pi)]^\circ$ denotes interior of $m(\pi)$ relative to M .

For $\lambda \in \Lambda(k, n)$, let $\nu^\lambda = \mu_+^\lambda - \mu_-^\lambda$ and define a k -vector valued measure over M by $\nu = \sum_{\lambda \in \Lambda(k, n)} \nu^\lambda e_\lambda$. Then, for any simple polyhedral region $\pi \subset X$,

$$\begin{aligned} |u(f^\lambda, \pi) - \nu^\lambda(m(\pi))| &\leq |u^+(f^\lambda, \pi) - \mu_+^\lambda(m(\pi))| + |u^-(f^\lambda, \pi) - \mu_-^\lambda(m(\pi))| \\ &\leq \mu^\lambda(m(\pi)) - |u(f^\lambda, \pi)|. \end{aligned}$$

Let θ^λ denote the Radon-Nikodym derivative of ν^λ with respect to μ for $\lambda \in \Lambda(k, n)$ and let $\theta = \sum_{\lambda \in \Lambda(k, n)} \theta^\lambda e_\lambda$. Then, by [3, 5ii], $|\theta(z)| = 1$ for μ almost every $z \in M$, where $|\cdot|$ denotes the Euclidean norm on $\bigwedge_k(R^n)$.

THEOREM 2. For $g \in C(M)$ and $\varphi \in E^k(R^n)$,

$$T(g)(\varphi) = \int_M g(z)\varphi(l(z)) \cdot \theta(z) d\mu.$$

Proof. The essential elements of a proof of this statement can be found in [3] and [13]. Because of the difference in viewpoint and notation, a short proof is included here.

Suppose $g \in C(M)$, $\varphi \in E^k(R^n)$, and $\varepsilon > 0$. Let $\psi(z) = g(z)\varphi(l(z))$ for $z \in M$. Then ψ is a continuous k -covector valued function on M . Since, for any polyhedral region $\pi \subset X$, the diameter of $m(\pi)$ does not exceed the diameter of $f(\pi)$ we have

$$\max_{\pi \in G} \sup \{ \|\psi(m(x)) - \psi(m(x'))\| : x', x \in \pi \} < \varepsilon$$

for $G \in \mathcal{G}$ with $\delta(G)$ sufficiently small.

For such a G consider

$$\Delta = \left| \sum_{\pi \in G} \frac{1}{|\pi|} \int_{\pi} \psi(m(x)) \cdot u(f, \pi) \, dx - \int_M \psi(z) \cdot \theta(z) \, d\mu \right|.$$

Since $\mu(m(\pi)) = \mu([m(\pi)]^o)$ for $\pi \in G$ we have

$$\begin{aligned} \Delta &\leq \sum_{\pi \in G} \left| \frac{1}{|\pi|} \int_{\pi} \psi(m(x)) \cdot u(f, \pi) \, dx - \int_{m(\pi)} \psi(z) \cdot \theta(z) \, d\mu \right| \\ &\quad + \left| \int_{M - \bigcup_{\pi \in G} m(\pi)} \psi(z) \cdot \theta(z) \, d\mu \right| \\ &\leq \varepsilon \sum_{\pi \in G} \|u(f, \pi)\| + M(\psi) \sum_{\pi \in G} \|u(f, \pi) - v(m(\pi))\| \\ &\quad + \varepsilon \sum_{\pi \in G} \|v(m(\pi))\| + M(\psi) \binom{n}{k} \mu \left(M - \bigcup_{\pi \in G} m(\pi) \right) \\ &\leq 2\varepsilon \binom{n}{k} V(f) + M(\psi) \sum_{\lambda \in \Lambda(k, n)} \sum_{\pi \in G} |u(f^\lambda, \pi) - v^\lambda(m(\pi))| \\ &\quad + M(\psi) \binom{n}{k} \left[\mu(M) - \sum_{\pi \in G} \mu(m(\pi)) \right] \\ &\leq 2\varepsilon \binom{n}{k} V(f) + M(\psi) \sum_{\lambda \in \Lambda(k, n)} \left[V(f^\lambda) - \sum_{\pi \in G} |u(f^\lambda, \pi)| \right] \\ &\quad + M(\psi) \binom{n}{k} \left[V(f) - \sum_{\pi \in G} |u(f, \pi)| \right], \end{aligned}$$

and the theorem follows.

By Lebesgue's bounded convergence theorem the conclusion of the above theorem remains valid if g is any bounded Borel measurable function on M .

COROLLARY. $\mu \leq \|T\| \leq \binom{n}{k} \mu$.

Proof. If B is a Borel set in M and $\varphi \in E^k(R^n)$, then

$$\begin{aligned} |T(B)(\varphi)| &= \left| \int_B \varphi(l(z)) \cdot \theta(z) \, d\mu \right| \\ &\leq M(\varphi) \int_B \|\theta(z)\| \, d\mu \leq M(\varphi) \binom{n}{k} \mu(B). \end{aligned}$$

Thus $\|T\| \leq \binom{n}{k} \mu$.

Suppose A is open in M . For each $G \in \mathcal{G}$ let

$$G(A) = G \cap \{\pi : \pi \subset m^{-1}(A)\}.$$

Then, by [13, §5.7],

$$\mu(A) = \lim_{\delta(G) \rightarrow 0} \sum_{\pi \in G(A)} |u(f, \pi)|.$$

Since

$$\begin{aligned} \sum_{\pi \in G} |u(f, \pi) - v(m(\pi))| &\leq \sum_{\lambda \in \Lambda(k, n)} \sum_{\pi \in G} |u(f^\lambda, \pi) - v^\lambda(m(\pi))| \\ &\leq \sum_{\lambda \in \Lambda(k, n)} \left[V(f^\lambda) - \sum_{\pi \in G} |u(f^\lambda, \pi)| \right] \leq \binom{n}{k} \delta(G), \end{aligned}$$

and $\mu(m(\pi)) = \mu([m(\pi)]^\circ)$ for $\pi \in G$, we have

$$\mu(A) = \lim_{\delta(G) \rightarrow 0} \sum_{\pi \in G(A)} |v(m(\pi))| \leq \limsup_{\delta(G) \rightarrow 0} \sum_{\pi \in G(A)} M(T([m(\pi)]^\circ)) \leq \|T\|(A).$$

If B is a Borel set in M and A is open with $B \subset A$ then

$$\mu(B) - \|T\|(B) \leq \mu(A) - \|T\|(B) \leq \|T\|(A - B) \leq \binom{n}{k} \mu(A - B).$$

3. Representation.

THEOREM 3. Suppose $f \in \mathcal{F}^*(k, n)$ and T is the current valued measure associated with f .

If there is a sequence $\{f_i\}$ of quasi-linear maps $f_i: X \rightarrow R^n$ converging uniformly to f with bounded areas and such that

$$T(g)(\varphi) = \lim_{i \rightarrow \infty} f_{i\#}[X \wedge (g \circ m)](\varphi)$$

for all $g \in C(M)$ and $\varphi \in E^k(R^n)$, then

1. T is rectifiable current valued.
2. There is a Baire function $v: M \rightarrow \bigwedge_k(R^n)$ such that, for $\|T\|$ almost every $z \in M$, $v(z)$ is a simple k -vector, $|v(z)|$ is an integer,

$$\varphi(l(z)) \cdot v(z) = \lim_{r \rightarrow 0+} \frac{T(\Delta(z, r))(\varphi)}{\alpha(k)r^k} \quad \text{for } \varphi \in E^k(R^n),$$

and

$$|v(z)| = \lim_{r \rightarrow 0+} \frac{\|T\|(\Delta(z, r))}{\alpha(k)r^k},$$

where $\alpha(k)$ is the k -dimensional Lebesgue measure of $\{x \in R^k : |x| \leq 1\}$ and, for $r > 0$, $\Delta(z, r)$ is the component of $l^{-1}(\{y : |y - l(z)| < r\})$ that contains z .

3. For each Borel set $B \subset M$,

$$\begin{aligned} T(B)(\varphi) &= \int_B \varphi(l(z)) \cdot v(z) dH^k_f \\ &= \int_{R^n} \varphi(y) \cdot \left\{ \sum_{z \in l^{-1}(y) \cap B} v(z) \right\} dH^n_k \quad \text{for } \varphi \in E^k(R^n), \end{aligned}$$

and

$$\|T\|(B) = \int_B |v(z)| dH_f^k = \int_{R^n} \left(\sum_{z \in I^{-1}(y) \cap B} |v(z)| \right) dH_n^k$$

where H_f^k and H_n^k denote k -dimensional Hausdorff measure in M and R^n respectively.

Proof. Let F denote the class of Borel sets $B \subset M$ for which $T(B)$ is rectifiable. Then F is closed under countable disjoint union and proper subtraction.

Using the arguments of [7, §§3.2 and 3.4] one shows that $A \in F$ whenever A is an open subset of M with $A \cap m(\text{Bdry } X) = \emptyset$. Since

$$\|T\|(m(\text{Bdry } X)) \leq \binom{n}{k} \mu(m(\text{Bdry } X)) = 0,$$

statement 1 follows.

From the hypothesis that, for $g \in C(M)$, $T(g)$ is the weak limit of the $f_{i\#}[X \wedge (g \circ m)]$ it follows readily that, for any open $V \subset M$ with $V \cap m(\text{Bdry } X) = \emptyset$, we have $\text{spt } \partial T(V) \subset I(\text{Bdry } V)$. This condition, together with $\|T\|(m(\text{Bdry } X)) = 0$ allows one to use, with obvious modifications, the arguments of [7, §§2.1 and 2.2] to prove statements 2 and 3.

We now investigate conditions under which the hypotheses of Theorem 3 are satisfied.

Suppose that f is a continuous mapping from a polyhedral region $X \subset R^k$ into R^k with $V(f) < \infty$.

For each polyhedral region $\sigma \subset X$ and $y \in R^k$ let

$$N(f, \sigma, y) = \sup \sum_{\pi \in S} |O(f, \pi, y)|$$

and

$$N^\pm(f, \sigma, y) = \sup \sum_{\pi \in S} O^\pm(f, \pi, y)$$

where the suprema are taken over all finite collections S of nonoverlapping simple polyhedral regions $\pi \subset \sigma$. Then for each σ the functions $N(f, \sigma, y)$, $N^\pm(f, \sigma, y)$ are nonnegative, integer valued, lower semicontinuous functions on R^k , $V(f|\sigma) = \int_{R^k} N(f, \sigma, y) dy$ and $N(f, \sigma, y) = N^+(f, \sigma, y) + N^-(f, \sigma, y)$ for almost all $y \in R^k$.

For each σ let

$$\begin{aligned} n(f, \sigma, y) &= N^+(f, \sigma, y) - N^-(f, \sigma, y) \quad \text{if } N(f, \sigma, y) < \infty, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

For each simple polyhedral region $\pi \subset X$ let

$$v(f, \pi) = \int_{R^k} |O(f, \pi, y)| dy.$$

LEMMA. Suppose $f: X \rightarrow R^k$ as above and $\{f_i\}$ is a sequence of continuous mappings $f_i: X \rightarrow R^k$ converging uniformly to f with $V(f_i) < \infty$ for $i = 1, 2, \dots$

If G is any finite collection of nonoverlapping simple polyhedral regions $\pi \subset X$, then

$$(a) \quad \limsup_{i \rightarrow \infty} \sum_{\pi \in G} |u(f, \pi) - u(f_i, \pi)| \leq 2 \left[V(f) - \sum_{\pi \in G} v(f, \pi) \right] + 2 \limsup_{i \rightarrow \infty} [V(f_i) - V(f)],$$

and

$$(b) \quad \limsup_{i \rightarrow \infty} \left[V(f_i) - \sum_{\pi \in G} v(f_i, \pi) \right] \leq V(f) - \sum_{\pi \in G} v(f, \pi) + \limsup_{i \rightarrow \infty} [V(f_i) - V(f)].$$

Proof. For $\pi \in G$, consider

$$\begin{aligned} |u(f, \pi) - u(f_i, \pi)| &\leq \left| u(f, \pi) - \int_{R^k} n(f, \pi, y) dy \right| \\ &\quad + \left| \int_{R^k} (n(f, \pi, y) - n(f_i, \pi, y)) dy \right| \\ &\quad + \left| \int_{R^k} n(f_i, \pi, y) dy - u(f_i, \pi) \right|. \end{aligned}$$

We have

$$\begin{aligned} \left| u(f, \pi) - \int_{R^k} n(f, \pi, y) dy \right| &= \left| \int_{R^k} (O(f, \pi, y) - n(f, \pi, y)) dy \right| \\ &\leq \int_{R^k} (N^+(f, \pi, y) - O^+(f, \pi, y)) dy + \int_{R^k} (N^-(f, \pi, y) - O^-(f, \pi, y)) dy \\ &= V(f|\pi) - v(f, \pi), \end{aligned}$$

and, similarly,

$$\left| u(f_i, \pi) - \int_{R^k} n(f_i, \pi, y) dy \right| \leq V(f_i|\pi) - v(f_i, \pi).$$

Let

$$B^\pm(\pi, i) = \{y \in R^k : N^\pm(f, \pi, y) > N^\pm(f_i, \pi, y)\}.$$

Then

$$\begin{aligned} &\left| \int_{R^k} (n(f, \pi, y) - n(f_i, \pi, y)) dy \right| \\ &\leq \int_{R^k} |N^+(f, \pi, y) - N^+(f_i, \pi, y)| dy + \int_{R^k} |N^-(f, \pi, y) - N^-(f_i, \pi, y)| dy \\ &\leq \int_{B^+(\pi, i)} N^+(f, \pi, y) dy + \int_{B^-(\pi, i)} N^-(f, \pi, y) dy \\ &\quad + \int_{R^k - B^+(\pi, i)} (N^+(f_i, \pi, y) - N^+(f, \pi, y)) dy \\ &\quad + \int_{R^k - B^-(\pi, i)} (N^-(f_i, \pi, y) - N^-(f, \pi, y)) dy \\ &\leq 2 \int_{B^+(\pi, i)} N^+(f, \pi, y) dy + 2 \int_{B^-(\pi, i)} N^-(f, \pi, y) dy + V(f_i|\pi) - V(f|\pi). \end{aligned}$$

Thus

$$\sum_{\pi \in G} |u(f, \pi) - u(f_i, \pi)| \leq V(f_i) - \sum_{\pi \in G} v(f, \pi) + V(f_i) - \sum_{\pi \in G} v(f_i, \pi) \\ + 2 \sum_{\pi \in G} \left\{ \int_{B^+(\pi, i)} N^+(f, \pi, y) dy + \int_{B^-(\pi, i)} N^-(f, \pi, y) dy \right\}.$$

By [12, §2.2], $v(f, \pi) \leq \liminf_{i \rightarrow \infty} v(f_i, \pi)$ and $N^\pm(f, \pi, y) \leq \liminf_{i \rightarrow \infty} N^\pm(f_i, \pi, y)$ for $\pi \in G$ and $y \in R^k$. Thus, given $\varepsilon > 0$, we have $\sum_{\pi \in G} v(f, \pi) - \varepsilon < \sum_{\pi \in G} v(f_i, \pi)$ for i sufficiently large. Hence

$$V(f_i) - \sum_{\pi \in G} v(f_i, \pi) < V(f_i) - V(f) + V(f) - \sum_{\pi \in G} v(f, \pi) + \varepsilon$$

and (b) is proved.

Now, for each $\pi \in G$, the functions $N^+(f, \pi, y)$ and $N^-(f, \pi, y)$ are integrable, the k -dimensional Lebesgue measure of $\bigcup_{i=1}^{\infty} B^\pm(\pi, i)$ is finite and

$$\bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} B^\pm(\pi, j) \subset \{y : N^\pm(f, \pi, y) = \infty\}.$$

Thus, since G is finite,

$$\limsup_{i \rightarrow \infty} \sum_{\pi \in G} |u(f, \pi) - u(f_i, \pi)| \leq 2 \left[V(f) - \sum_{\pi \in G} v(f, \pi) \right] + 2 \limsup_{i \rightarrow \infty} [V(f_i) - V(f)]$$

and (a) is proved.

THEOREM 4. Suppose f is a continuous mapping of a polyhedral region $X \subset R^k$ into R^n with $f \in \mathcal{F}^*(k, n)$ and let T denote the current valued measure associated with f . If $\{f_i\}$ is a sequence of quasi-linear maps $f_i: X \rightarrow R^n$ converging uniformly to f with $V(f^\lambda) = \lim_{i \rightarrow \infty} V(f_i^\lambda)$ for $\lambda \in \Lambda(k, n)$, then

$$T(g)(\varphi) = \lim_{i \rightarrow \infty} f_{i\#}[X \wedge (g \circ m)](\varphi)$$

for all $g \in C(M)$ and $\varphi \in E^k(R^n)$.

Proof. Suppose $g \in C(M)$, $\varphi \in E^k(R^n)$, and $\varepsilon > 0$. Note that

$$f_{i\#}[X \wedge (g \circ m)](\varphi) = \int_X g(m(x)) \varphi(f_i(x)) \cdot Jf_i(x) dx$$

where $Jf_i(x) = \sum_{\lambda \in \Lambda(k, n)} Jf_i^\lambda(x) e_\lambda$. Here Jf_i^λ denotes the ordinary Jacobian of f_i^λ . Also, from [1, 8.9iii] we have $u(f_i^\lambda, \pi) = \int_\pi Jf_i^\lambda(x) dx$ whenever π is a simple polyhedral region in X .

Let $G \in \mathcal{G}$ with $\delta(G)$ so small that

$$\left| T(g)(\varphi) - \sum_{\pi \in G} \frac{1}{|\pi|} \int_\pi g(m(x)) \varphi(f(x)) \cdot u(f, \pi) dx \right| < \varepsilon,$$

and

$$\max \{|g(m(x)) \varphi(f(x)) - g(m(x')) \varphi(f(x'))| : x, x' \in \pi\} < \varepsilon$$

for $\pi \in G$. Then

$$\begin{aligned}
 & \left| \sum_{\pi \in G} \frac{1}{|\pi|} \int_{\pi} g(m(x)) \varphi(f(x)) \cdot u(f, \pi) \, dx - \int_X g(m(x)) \varphi(f_i(x)) \cdot Jf_i(x) \, dx \right| \\
 & \leq \sum_{\pi \in G} \left| \frac{1}{|\pi|} \int_{\pi} g(m(x)) \varphi(f(x)) \cdot [u(f, \pi) - u(f_i, \pi)] \, dx \right| \\
 & \quad + \sum_{\pi \in G} \left| \int_{\pi} g(m(x)) \varphi(f(x)) \cdot \left[\frac{u(f_i, \pi)}{|\pi|} - Jf_i(x) \right] \, dx \right| \\
 & \quad + \sum_{\pi \in G} \left| \int_{\pi} g(m(x)) [\varphi(f(x)) - \varphi(f_i(x))] \cdot Jf_i(x) \, dx \right| \\
 & \quad + \left| \int_{X - \bigcup_{\pi \in G} \pi} g(m(x)) \varphi(f_i(x)) \cdot Jf_i(x) \, dx \right| \\
 & \leq M(g)M(\varphi \circ f) \sum_{\pi \in G} \|u(f, \pi) - u(f_i, \pi)\| + \varepsilon \sum_{\pi \in G} \left\{ |u(f_i, \pi)| + \int_{\pi} |Jf_i(x)| \, dx \right\} \\
 & \quad + M(g)M(\varphi \circ f - \varphi \circ f_i) \sum_{\pi \in G} V(f_i|\pi) + M(g)M(\varphi \circ f_i) V\left(f_i|X - \bigcup_{\pi \in G} \pi\right) \\
 & \leq M(g)M(\varphi \circ f) \sum_{\lambda \in \Lambda(k, n)} \sum_{\pi \in G} |u(f^{\lambda}, \pi) - u(f_i^{\lambda}, \pi)| + 2\varepsilon V(f_i) \\
 & \quad + M(g)M(\varphi \circ f - \varphi \circ f_i) V(f_i) + M(g)M(\varphi \circ f_i) \sum_{\lambda \in \Lambda(k, n)} \left[V(f_i^{\lambda}) - \sum_{\pi \in G} v(f_i^{\lambda}, \pi) \right].
 \end{aligned}$$

By the preceding lemma we have

$$\limsup_{i \rightarrow \infty} \sum_{\pi \in G} |u(f^{\lambda}, \pi) - u(f_i^{\lambda}, \pi)| \leq 2[V(f^{\lambda}) - \sum_{\pi \in G} |u(f^{\lambda}, \pi)|] < 2\delta(G),$$

and

$$\limsup_{i \rightarrow \infty} \left[V(f_i^{\lambda}) - \sum_{\pi \in G} v(f_i^{\lambda}, \pi) \right] \leq V(f^{\lambda}) - \sum_{\pi \in G} |u(f^{\lambda}, \pi)| < \delta(G)$$

for $\lambda \in \Lambda(k, n)$.

Since $\{f_i\}$ converges uniformly to f we have

$$\lim_{i \rightarrow \infty} M(\varphi \circ f - \varphi \circ f_i) = 0$$

and the theorem follows.

REMARK. Theorem 4 was originally proved under the hypothesis that given any $\varepsilon > 0$ there is a $G \in \mathcal{G}$ such that

$$\delta(G) + \max \left\{ \mathcal{L}_k \left(f^{\lambda} \left(\bigcup_{\pi \in G} \text{Bdry } \pi \right) \right) : \lambda \in \Lambda(k, n) \right\} < \varepsilon.$$

Here $\mathcal{L}_k$ denotes k -dimensional Lebesgue measure in R^k . It was shown in [11, §3] that this hypothesis is satisfied if either $k=2$ or $H_n^{k+1}(f(X))=0$.

The proof of Theorem 4 under the weaker hypothesis $f \in \mathcal{F}^*(k, n)$ was suggested by T. Nishiura.

THEOREM 5. *If f is a continuous mapping from a polyhedral region $X \subset R^k$ into R^n , $2 \leq k \leq n$, $V(f) < \infty$, and either $k=2$ or $H_n^{k+1}(f(X))=0$, then $f \in \mathcal{T}^*(k, n)$ and there is a sequence $\{f_i\}$ of quasi-linear mappings $f_i: X \rightarrow R^n$ converging uniformly to f with $V(f^\wedge) = \lim_{i \rightarrow \infty} V(f_i^\wedge)$ for $\lambda \in \Lambda(k, n)$.*

Proof. That, under these conditions, $f \in \mathcal{T}^*(k, n)$ was proved in [11]. The proofs of [4, Theorems 3.16 and 5.7] consist of constructions, under the given conditions, of sequences of quasi-linear maps having the desired properties.

4. Densities. Throughout this section assume that $f: X \rightarrow R^n$ satisfies the hypotheses of Theorem 3. We will show that $\mu = \|T\|$ and describe the relation between the functions v and θ .

For each $z \in M$, let

$$\mu'(z) = \limsup_{r \rightarrow 0+} \frac{\mu(\Delta(z, r))}{\alpha(k)r^k}.$$

It is readily shown that μ' is Borel measurable.

THEOREM 6. *For each Borel set $B \subset M$, $\int_B \mu'(z) dH_f^k \leq \mu(B)$.*

Proof. Using the arguments of [5, Lemma 6.1] and the definition of μ we find that, if A is a Borel subset of M and $A \subset \{z : \mu'(z) > c\}$, $c > 0$, then $\mu(A) \geq cH_f^k(A)$. Adapting the proof of [5, Theorem 6.2], the theorem follows.

According to Theorem 3, for $\|T\|$ almost every $z \in M$, there is a simple k -covector ω with $|\omega| = 1$ such that

$$|v(z)| = \omega \cdot v(z) = \lim_{r \rightarrow 0+} \frac{T(\Delta(z, r))(\omega)}{\alpha(k)r^k}.$$

For $r > 0$,

$$T(\Delta(z, r))(\omega) = \int_{\Delta(z, r)} \omega \cdot \theta(z) d\mu \leq \int_{\Delta(z, r)} |\omega| |\theta(z)| d\mu = \mu(\Delta(z, r)).$$

Thus, for $\|T\|$ almost every $z \in M$, $|v(z)| \leq \mu'(z)$ and, hence, for any Borel set $B \subset M$,

$$\|T\|(B) = \int_B |v(z)| dH_f^k \leq \int_B \mu'(z) dH_f^k \leq \mu(B).$$

By the corollary to Theorem 2, $\mu = \|T\|$.

Thus, by Theorem 3,

$$1 \leq \mu'(z) = \lim_{r \rightarrow 0+} \frac{\mu(\Delta(z, r))}{\alpha(k)r^k} < \infty$$

for μ almost every $z \in M$, and, hence

$$\lim_{r \rightarrow 0+} \frac{\mu(\Delta(z, 5r))}{\mu(\Delta(z, r))} \leq 5^k$$

for μ almost every $z \in M$.

For $z \in M$ and $r > 0$, let $c(z, r)$ denote the closure of $\Delta(z, r)$ in M and let $V = \{c(z, r) : z \in M, 0 < r < 1, \mu(\text{Bdry } c(z, r)) = 0,$

$$\text{and } \mu(c(z, 5r)) < (5^k + 1)\mu(\Delta(z, r))\}.$$

Note that $\inf \{r : c(z, r) \in V\} = 0$ for μ almost every $z \in M$.

For $S \subset M$, let $\text{diam}_f(S)$ denote the diameter of S in M , and, for $S \in V$, let

$$\hat{S} = \bigcup \{S' : S' \in V, S' \cap S \neq \emptyset, \text{ and } \text{diam}_f(S') \leq \frac{4}{3} \text{diam}_f(S)\}.$$

If $S = c(z, r)$, $S' = c(z', r')$, $S \cap S' \neq \emptyset$, and $\text{diam}_f(S') \leq \frac{4}{3} \text{diam}_f(S)$, then, for any $z'' \in c(z', r')$, we have

$$|l(z'') - l(z)| \leq \text{diam}_f(S') + \text{diam}_f(S) \leq (1 + \frac{4}{3})(2r) < 5r,$$

and hence, since S' is connected, $S' \subset c(z, 5r)$.

Thus, for $S \in V$, we have $\mu(\hat{S}) < (5^k + 1)\mu(S)$. Referring to [8, Theorem 2.8.7] we have the following

LEMMA. Suppose A is a Borel set in M , W is open in M , $F \subset V$, and

$$\inf \{r : c(z, r) \in F\} = 0$$

for all $z \in A$. Then there exists a countable disjointed subfamily G of F such that

$$\bigcup_{S \in G} S \subset W \quad \text{and} \quad \mu\left(A \cap W - \bigcup_{S \in G} S\right) = 0.$$

In view of this lemma and [8, Theorem 2.9.7] we find that, for $\lambda \in \Lambda(k, n)$,

$$\gamma_{\pm}^{\lambda}(z) = \lim_{r \rightarrow 0+} \frac{\mu_{\pm}^{\lambda}(\Delta(z, r))}{\mu(\Delta(z, r))}$$

exists and is finite for μ almost every $z \in M$, and since $\mu_{\pm}^{\lambda} \leq \mu$,

$$\mu_{\pm}^{\lambda}(B) = \int_B \gamma_{\pm}^{\lambda}(z) d\mu$$

for each Borel set $B \subset M$.

Thus,

$$\theta^{\lambda}(z) = \gamma_{+}^{\lambda}(z) - \gamma_{-}^{\lambda}(z) = \lim_{r \rightarrow 0+} \frac{\nu^{\lambda}(\Delta(z, r))}{\mu(\Delta(z, r))}$$

for μ almost every $z \in M$.

We summarize the results of this section in

THEOREM 7. If $f: X \rightarrow R^n$ satisfies the hypotheses of Theorem 3, then $\mu = \|T\|$ and $v(z) = \mu'(z)\theta(z)$ for μ almost every $z \in M$.

Proof. Let $\{e^{\lambda} : \lambda \in \Lambda(k, n)\}$ denote the basis of $\bigwedge^k(R^n)$ dual to $\{e_{\lambda} : \lambda \in \Lambda(k, n)\}$.

For each $\lambda \in \Lambda(k, n)$,

$$e^{\lambda} \cdot v(z) = \lim_{r \rightarrow 0+} \frac{T(\Delta(z, r))(e^{\lambda})}{\alpha(k)r^k}$$

for μ almost every $z \in M$.

Now, by Theorem 2,

$$T(\Delta(z, r))(e^\lambda) = \int_{\Delta(z, r)} e^\lambda \cdot \theta(z) \, d\mu = \nu^\lambda(\Delta(z, r)),$$

and, hence,

$$e^\lambda \cdot \nu(z) = \lim_{r \rightarrow 0+} \frac{\nu^\lambda(\Delta(z, r))}{\alpha(k)r^k} = \mu'(z)\theta^\lambda(z).$$

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